

# BRANS-DICKE WORMHOLE REVISITED-II

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## Abstract

In a recent paper, a wormhole range in the Jordan frame,  $-3/2 < \omega < -4/3$  for the vacuum Brans-Dicke Class I solution was derived. On general grounds and under certain conditions, it is shown in a theorem that static wormhole solutions in the scalar-tensor theory are not possible. We agree with the conclusion within its framework but report that a singularity-free wormhole *can* be obtained from Class I solution by performing certain operations on it, a fact possibly not yet widely known. The transformed solution is regular everywhere, produces a wormhole with two asymptotically flat regions for a revised new range  $-2 < \omega < -3/2$ , together with a wormhole analogue (of Horowitz-Ross naked black hole) that we discovered earlier. This new range lies in the ghost regime in the Einstein frame consistent with the theorem. We further conclude that there is a fixed point at  $\omega = -3/2$ , the values  $\omega > -3/2$  correspond to singular wormholes, while values  $\omega < -3/2$  correspond to singularity-free wormholes.

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## I. Introduction

Lorentzian wormholes as possible astrophysical objects has been under active investigation for quite some time now. In particular, the possibility of occurrence of such objects in the Brans-Dicke theory is quite intriguing since it is a natural theory that emerged as a Machian alternative to Einstein's theory of general relativity. To our knowledge, a theoretical search for static wormholes in the vacuum Brans-Dicke theory has been initiated by Agnese and La Camera [1] who have shown that the Brans-Dicke scalar  $\varphi$  can play the role of

exotic matter provided the coupling parameter  $\omega < -2$ , followed by the work of Visser and Hochberg [2], and by the works in other classes of Brans solutions [3] in the Jordan Frame (JF or  $\mathbf{M}_J$ ) as well as in the conformally rescaled Einstein Frame (EF or  $\mathbf{M}_E$ ) [4]. There exist hundreds of articles on wormholes today, but we only mention some works on Brans-Dicke wormholes [5-15]. A particularly interesting recent result is that the Horowitz-Ross naked black hole has a wormhole analogue in the Brans Class I solution [13]. Considering the importance of Brans-Dicke theory in the interpretation of various astrophysical phenomena, it is important that certain questions raised in the literature relating to static spherically symmetric wormhole solutions in the vacuum theory be clarified. The purpose of the present paper is to do that.

For static spherically symmetric Class I solution in the  $\omega = \text{constant}$  vacuum Brans-Dicke theory, we proposed a wormhole range  $-3/2 < \omega < -4/3$  in the JF and developed a wormhole analogy to Horowitz-Ross naked black holes [13] for  $\omega < -2$ , the range obtained previously by Agnese and La Camera [1]. Against this proposal, we identified two criticisms: (A) The range  $-3/2 < \omega < -4/3$  lies in the so called "no-ghost" regime in EF<sup>1</sup> and (B) The occurrence of naked singularity in the Class I solution spoils the spacetime [15, p.5]. On general grounds, it is proved in a theorem [14, pp.3,6] that, under certain conditions, there is no wormhole connecting two spatial infinities in the scalar-tensor theory. In [14,15], it is of course implied, and we accept, that a true wormhole is meant to be a non-singular one with two asymptotically flat regions connected by a throat. A singular wormhole is "diseased" [16]; in fact it is long recognized that the Class I solution is plagued by such curvature singularity<sup>2</sup>. In this work, we are concerned only with the special case of Class I solution in the  $\omega = \text{constant}$  vacuum Brans-Dicke theory.

We agree with the theorem within the framework it is proved. When applied to the special case under consideration, its message may be summarized in the paradigm "no ghost matter  $\Rightarrow$  no wormhole" in either frame connected by conformal mapping [14, p.3]. However, conformal mapping does *not* always guarantee that ghost-free matter in EF corresponds to ghost-free matter in JF and vice versa. Our earlier work in fact showed that the ghost-free regime  $\omega > -3/2$  in EF can still lead to ghost-matter in JF [13] together with the existence of a real throat and twice asymptotically flat regions (inversion invariant under  $r \rightarrow B^2/r$ ). In spite of these features, we understand that Class I solution does not qualify as a true wormhole. One may say that  $r = 0$  flat spatial infinity is divided from the one at  $r = \infty$  by a curvature singularity at  $r = B$ . So, the inversion transformation in fact relates two separate singular space-times, not

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<sup>1</sup>Private correspondence based on [14]. By ghost matter, we mean here matter that violates energy conditions, at least the Null Energy Condition (NEC). The values of  $\omega$  in the scalar field redefinition (4) determine the sign of the resulting kinetic term in EF action. When  $\alpha < 0$ ,  $\omega < -3/2$ , the sign is negative and  $\omega$  is said to be in the ghost regime. See Sec.II.

<sup>2</sup>Professor Starobinsky, in another correspondence, has pointed out an additional problem: The existence of a throat for any value of  $\omega > -3/2$  does not prevent the formation of a curvature singularity behind it. The difference between the  $\omega > -4/3$  and  $\omega < -4/3$  cases is only that the singularity lies at finite or infinite proper radial distance from the throat, but there is no asymptotic flatness (no "another universe") behind the throat in both cases.

two regions of one *connected* space-time required of a true wormhole. Having said this, we argue that the theories in JF and EF are two fundamentally distinct theories and hence it seems more logical to draw conclusions about wormhole  $\omega$ -regime in JF from the geometry in JF itself.

The curvature singularity in Class I solution (in its usual form) is well known. We show here that by some operations the maladies in the Class I solution can be redressed leading to an asymptotically flat, singularity-free wormhole in the new range  $-2 < \omega < -3/2$ . We also show that a gauge non-uniqueness of solutions allow us to shift the values  $\omega$  on either side of the divide  $\omega = -3/2$  in the vacuum JF Brans-Dicke theory. We take  $16\pi G = c = 1$  and a signature convention  $(-, +, +, +)$ .

## II. Two distinct theories

For the ease of argument, we restate the actions of two distinct theories. First is the action of the vacuum Brans-Dicke theory in the JF  $(g_{\mu\nu}, \varphi)$ :

$$S = \int d^4x (-g)^{\frac{1}{2}} [\varphi \mathbf{R} + \omega \varphi^{-1} g^{\mu\nu} \varphi_{,\mu} \varphi_{,\nu}]. \quad (1)$$

The field equations are

$$\square^2 \varphi = 0,$$

$$\mathbf{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathbf{R} = -\frac{\omega}{\varphi^2} \left[ \varphi_{,\mu} \varphi_{,\nu} - \frac{1}{2} g_{\mu\nu} \varphi_{,\sigma} \varphi^{,\sigma} \right] - \frac{1}{\varphi} [\varphi_{,\mu;\nu} - g_{\mu\nu} \square^2 \varphi], \quad (2)$$

where  $\square^2 \equiv (\varphi^{;\rho})_{;\rho}$  and  $\omega = \text{constant}$  is a dimensionless coupling parameter.

Consider the conformal transformation

$$\tilde{g}_{\mu\nu} = \varphi g_{\mu\nu}, \quad (3)$$

and a redefinition of the Brans-Dicke scalar  $\varphi \rightarrow \phi$  by

$$d\phi = \left( \frac{\omega + 3/2}{\alpha} \right)^{1/2} \frac{d\varphi}{\varphi}, \quad (4)$$

in which we have intentionally introduced an arbitrary constant parameter  $\alpha$ . These transformations are known for long in the literature as Dicke transformations. The resulting action in the EF  $(\tilde{g}_{\mu\nu}, \phi)$  is

$$S = \int d^4x (-\tilde{g})^{1/2} [\tilde{\mathbf{R}} + \alpha \tilde{g}^{\mu\nu} \phi_{,\mu} \phi_{,\nu}]. \quad (5)$$

This is the Hilbert-Einstein action of Einstein's general relativity with a source kinetic term  $\alpha \tilde{g}^{\mu\nu} \phi_{,\mu} \phi_{,\nu}$  that leads to Einstein minimally coupled field equations. As one sees,  $\omega$  has disappeared from the action. To ensure ghost matter needed for wormholes, one has to have a *negative* kinetic term. The common prescription is to assume  $\alpha < 0$  and a real  $\phi$  in EF, which immediately yield

from definition (4) the ghost regime  $\omega < -3/2$ . This regime naturally is in contradiction to the range  $-3/2 < \omega < -4/3$  derived in [13].

We wish to state the following (for the definition of real constants  $C$ ,  $\omega$  and  $\lambda$  referred to below, see next section):

(1) Visser and Hochberg [2] detailed the geometry of vacuum Brans Class I wormhole in JF for different ranges of  $\omega$  ( $\omega < -2$  as well as  $\omega > -3/2$ ). We want to emphasize that the new finite regime that we derived and studied ( $-3/2 < \omega < -4/3$ ) is a result of our JF wormhole condition  $(C+1)^2 > \lambda^2$ , which is *weaker* than conditions on  $B_{\text{VH}} = -\frac{C+1}{\lambda}$ . To be more precise, Visser and Hochberg [2] used only  $\lambda > 0$ , whereas we permitted  $\lambda$  to have both signs allowing us to carve out a narrow, but new, regime for  $\omega$  at the expense of sacrificing the general relativity limit at  $\omega \rightarrow \pm\infty$ . (It is not even any sacrifice, as is known today, see last few references cited in [13] and Sec.IV below).

(2) What is the basis for our derived range  $-3/2 < \omega < -4/3$ ? Note that from most general considerations, the weak field value of  $C$  is  $C = -\frac{1}{\omega+2}$ , which yields  $\lambda = \pm\sqrt{\frac{2\omega+3}{2\omega+4}}$ . The *reality* of  $\lambda$  suggests that  $-3/2 < \omega$ . The other limit,  $\omega < -4/3$ , is obtained by imposing on the solution the fundamental wormhole constraint  $(C+1)^2 > \lambda^2$  and requiring further that the throat be real. Nowhere was it necessary to appeal to the transformations (3),(4). The parameters we used in [13] were all from within the JF theory, where  $\omega$  is understood with its full physical meaning. The singularity at  $r = B$  was only too evident, but we relied on the fact that the traveler had no access to it.

(3) The no-wormhole theorem is based on mapping between JF and EF via conformal transformations whereas we know that conformally related geometries are not the same geometries; for instance, curved cosmological metrics can be conformally flat. There are other reasons too, some of which are well known: (i) There is ostensibly no  $\omega$  in the minimally coupled EF action, a theory that can stand independently by itself without any umbilical relation to JF. (ii) The minimally coupled theory is *no* Machian Brans-Dicke scalar-tensor theory, but already the non-Machian Einstein's general relativity with a material source term. (iii) The vacuum JF theory is *conformally invariant* while EF theory is not (See Sec.IV). (iv) Contrary to prevailing belief, it is impossible to obtain Einstein's theory in the  $|\omega| \rightarrow \infty$  limit of the vacuum Brans-Dicke theory [17]. (v) The kinetic term (giving ghost or no-ghost) is identifiable as a source only in the EF action (5), while it is unidentifiable in the JF action (1) and finally (vi) The two frames can also be observationally distinguishable [18].

Despite the above, some authors consider the Einstein minimally coupled theory (5) exactly the same as Brans-Dicke theory (1), only rephrased in the Einstein frame. We disagree with this point of view. Some authors regard the EF as a convenient mathematical tool to draw conclusions about JF. While this procedure could possibly be useful for stability analysis, it could be misleading as well. For instance, the conditions  $\alpha > 0$ ,  $\omega > -3/2$ ,  $\phi$  real lead to a positive sign kinetic term (i.e., no ghost, no energy condition violation) in the EF action (5), but they would hide the fact that in JF, nonetheless, the energy conditions are violated, real throats do exist for  $\omega > -3/2$  in the Class I solution. But the

hope for a true wormhole is dashed by the occurrence of singularities both in JF and EF version of the solution. The purpose of these arguments is to make a case for the principle that one should look for true wormhole  $\omega$ -regime staying in any one frame, and not make a detour into the other, when drawing physical conclusions about solutions in that frame.

We next turn to criticism (B) that has in fact prodded us to look for regular wormholes in the JF, if any. As mentioned before, the appearance of naked singularity in Brans Class I spacetime is a genuine problem because it does not make the solution in its usual form look like a wormhole worth its name. We resolve the problem by certain explicit operations on the solution, as shown below.

### III. Brans Class I solution

The general solution of the field equations (2), in isotropic coordinates  $(t, r, \theta, \psi)$ , is given by

$$d\tau^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} [dr^2 + r^2(d\theta^2 + \sin^2 \theta d\psi^2)]. \quad (6)$$

Brans class I solution [19] in JF is given by

$$e^{\alpha(r)} = e^{\alpha_0} \left[ \frac{1 - B/r}{1 + B/r} \right]^{\frac{1}{\lambda}}, \quad (7)$$

$$e^{\beta(r)} = e^{\beta_0} [1 + B/r]^2 \left[ \frac{1 - B/r}{1 + B/r} \right]^{\frac{\lambda - C - 1}{\lambda}}, \quad (8)$$

$$\varphi(r) = \varphi_0 \left[ \frac{1 - B/r}{1 + B/r} \right]^{\frac{C}{\lambda}}, \quad (9)$$

$$\lambda^2 \equiv (C + 1)^2 - C \left( 1 - \frac{\omega C}{2} \right) > 0, \quad (10)$$

where  $\lambda$ ,  $\alpha_0$ ,  $\beta_0$ ,  $B$ ,  $C$ , and  $\varphi_0$  are real constants. The constants  $\alpha_0$  and  $\beta_0$  are determined by asymptotic flatness at  $r = \infty$  as  $\alpha_0 = \beta_0 = 0$ .

To see the naked singularity at  $r = B$ , it is enough to consider the invariant Riemann curvature component in the freely falling orthonormal frame  $(\hat{e}_0, \hat{e}_1, \hat{e}_2, \hat{e}_3)$  (See [13] for details)

$$\mathbf{R}_{\hat{1}\hat{0}\hat{1}\hat{0}} = \frac{4Br^3 Z^2 [\lambda(r^2 + B^2) - Br(C + 2)]}{\lambda^2 (r^2 - B^2)^4}, \quad (11)$$

where

$$Z \equiv \left( \frac{r - B}{r + B} \right)^{(C+1)/\lambda}. \quad (12)$$

Clearly,  $\mathbf{R}_{\hat{1}\hat{0}\hat{1}\hat{0}} \rightarrow \infty$  as  $r \rightarrow B$ . All curvature invariants also exhibit this behavior. To remove this singularity, we do the following operations on the Class I solution [21]:

$$r \rightarrow \frac{1}{r'}, B \rightarrow \frac{i}{B'}, \lambda \rightarrow -i\Lambda, \alpha_0 \rightarrow \epsilon_0, \beta_0 \rightarrow \zeta_0 + 2 \ln B', \quad (13)$$

where  $B', \Lambda$  are real. Using the identity

$$\tan^{-1}(x) = \frac{i}{2} \ln \left( \frac{1-ix}{1+ix} \right), \quad (14)$$

we arrive at the metric functions and the scalar field as follows

$$d\tau^2 = -e^{2\alpha(r')} dt^2 + e^{2\beta(r')} [dr'^2 + r'^2 (d\theta^2 + \sin^2 \theta d\psi^2)], \quad (15)$$

where

$$\alpha(r') = \epsilon_0 + \frac{2}{\Lambda} \tan^{-1} \left( \frac{r'}{B'} \right) \quad (16)$$

$$\beta(r') = \zeta_0 - \frac{2(C+1)}{\Lambda} \tan^{-1} \left( \frac{r'}{B'} \right) - \ln \left( \frac{r'^2}{r'^2 + B'^2} \right) \quad (17)$$

$$\varphi(r') = \varphi_0 \exp \left[ \frac{2C}{\Lambda} \tan^{-1} \left( \frac{r'}{B'} \right) \right] \quad (18)$$

$$\Lambda^2 \equiv C \left( 1 - \frac{\omega C}{2} \right) - (C+1)^2 > 0. \quad (19)$$

Asymptotic flatness requires that  $\epsilon_0 = -\frac{\pi}{\Lambda}$  and  $\zeta_0 = \frac{\pi(C+1)}{\Lambda}$ . This form of Class I solution has been listed by Brans [19] as his Class II solution, but we see that the two classes are *not* independent – one can be derived from the other by Wick rotation. Likewise, we show that the singular Ellis I and the non-singular Ellis III solutions in the minimally coupled theory are not independent solutions (See Appendix for an outline). *The main conclusion is that the solution set (15)-(19) is the counterpart in JF of the well known Ellis III wormhole in EF.*

The solution set (15)-(19) is regular everywhere including at  $r' = B'$  as can be verified by computing the curvature invariants. It has two asymptotically flat regions<sup>3</sup> with two asymmetric masses  $\frac{2B'C}{\Lambda}$  and  $-\frac{2B'C}{\Lambda} \exp[-\frac{\pi}{\Lambda}]$  on either side connected by a throat at

$$r_0^{\pm} = B' \left[ \frac{C+1}{\Lambda} \pm \sqrt{1 + \left( \frac{C+1}{\Lambda} \right)^2} \right], \quad (20)$$

defined by the minimum of the area radius  $r' \exp[\beta(r')]$ . The Riemann curvature component invariant under Lorentz boost is

$$\mathbf{R}_{\hat{1}\hat{0}'\hat{1}\hat{0}'} = -\frac{4B'^5 r'^3 [\Lambda B'^2 + B' r' (C+2) - \Lambda r'^2]}{\Lambda^2 (r'^2 + B'^2)^4} \exp \left[ \frac{8(C+1)}{\Lambda} \tan^{-1} \left( \frac{r'}{B'} \right) \right], \quad (21)$$

which is finite everywhere, and  $\mathbf{R}_{\hat{1}\hat{0}'\hat{1}\hat{0}'} \rightarrow 0$  as  $r' \rightarrow \pm\infty$ . All the curvature invariants are also finite and go to zero as  $r' \rightarrow \pm\infty$ . These facts resolve the maladies associated with the original form of Class I solution.

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<sup>3</sup>But not inversion invariant due to asymmetry, like in the Ellis III wormhole.

Using the weak field value  $C = -\frac{1}{\omega+2}$ , we get

$$\Lambda = \pm \sqrt{-\frac{2\omega+3}{2\omega+4}}. \quad (22)$$

To analyze the behavior of curvature  $\mathbf{R}_{\hat{1}'\hat{0}'\hat{1}'\hat{0}'}$  or the radial tidal force, we first implement that  $\Lambda$  be real, which immediately yields a new range  $-2 < \omega < -3/2$ . Putting the values of  $C$  and  $\Lambda$ , together with any value of  $\omega$  in the said range, we would obtain two values for the throat  $r_0'^{\pm}$ , one positive and the other negative, for real  $B'$ . Next, we shall discard the negative value for  $r_0'$  due to the fact that it would correspond to negative circumferential radius  $2\pi R_0$  for the throat defined in the generic Morris-Thorne form<sup>4</sup>:

$$R_0 = r_0' \left[ 1 + \frac{B'^2}{r_0'^2} \right] \exp \left[ \zeta_0 - \frac{2(C+1)}{\Lambda} \tan^{-1} \left( \frac{r_0'}{B'} \right) \right]. \quad (23)$$

Putting the value of  $C$  and either value of  $\Lambda$  in turn we can express  $\mathbf{R}_{\hat{1}'\hat{0}'\hat{1}'\hat{0}'} = g(\omega, r', B')$ , where the function  $g$  results from the right hand side of (21). Finally, the behavior of  $g$  in the figures 1 and 2 exhibit the wormhole analogue of the naked black hole. For positive  $\Lambda$ , curvature increases, while for negative  $\Lambda$ , curvature depletes above the throat, as measured by a Lorentz boosted observer. In either case, the hump and dip in the plots show that the curvature function  $g$  does not monotonically increase or decrease near the throat, which resemble the phenomena occurring near the horizon in naked black holes.

#### IV. Conformal invariance of vacuum Brans-Dicke theory

We point out an important fact about JF vacuum Brans-Dicke theory. Under conformal transformations [17,20,22]

$$\tilde{g}_{\mu\nu} = \varphi^{2\xi} g_{\mu\nu} \quad (24)$$

and a redefinition of the scalar

$$\sigma = \varphi^{1-2\xi}, \quad (25)$$

the Brans-Dicke action (1) remains invariant

$$S = \int d^4x (-\tilde{g})^{1/2} \left[ \sigma \tilde{\mathbf{R}} + \tilde{\omega} \sigma^{-1} \tilde{g}^{\mu\nu} \sigma_{,\mu} \sigma_{,\nu} \right], \quad (26)$$

where  $\xi$  is a real gauge parameter and

$$\tilde{\omega} = \frac{\omega - 6\xi(\xi - 1)}{(1 - 2\xi)^2}. \quad (27)$$

The invariance means that the vacuum solutions are not unique. Given any solution  $(g_{\mu\nu}, \varphi, \omega)$ , it is possible to generate any other solution  $(\tilde{g}_{\mu\nu}, \sigma, \tilde{\omega})$ .

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<sup>4</sup>The radial coordinate  $r'$  is an abstract coordinate chart covering the entire space, whereas the Morris-Thorne radius  $R$  is defined by physically measurable circumference but it does not cover the entire space. The throat radius can be calculated either by the minimal area radius involving  $r'$  or from the shape function involving  $R$ . Both of course yield the same answer.

The transformations (24), (25), collectively denoted by  $T_\xi$ , form a 1-parameter Abelian group for  $\xi \neq \frac{1}{2}$  and all the Brans-Dicke manifolds  $(g_{\mu\nu}, \varphi, \omega)$  mapped by  $T_\xi$  into  $(\tilde{g}_{\mu\nu}, \sigma, \tilde{\omega})$  form an equivalence class  $\mathcal{F}$ . The effect of  $T_\xi$  is to move  $\omega$  into another value  $\tilde{\omega}$  depending on the choice of gauge  $\xi$ , except the identity transformation  $\omega = \tilde{\omega}$  at  $\xi = 0, 1$ . *The  $\omega \rightarrow \infty$  limit can also be seen as a parameter change that moves Brans-Dicke theory within the same class  $\mathcal{F}$ , and therefore it cannot reproduce Einstein's general relativity, which does not belong to  $\mathcal{F}$ .* The conformal invariance is broken when ordinary matter is added to the action except when the trace  $\mathbf{T}^{(\text{matter})} = 0$ . When  $\xi \rightarrow \frac{1}{2}$ , one has  $\tilde{\omega} \rightarrow \infty$ , the behavior of  $\sigma$  becomes problematic<sup>5</sup>. One could instead use the redefinition (4) but then the parameter  $\omega$  disappears altogether in (5), and the  $\omega \rightarrow \infty$  limit cannot be considered because the theory is already GR, apart from a possible violation of the equivalence principle due to the anomalous coupling of the scalar to the energy-momentum tensor of ordinary matter, if  $T_{\mu\nu} \neq 0$ . It is evident from (27) that a fixed finite value of  $\omega$  is not sacrosanct – it can be moved to any desired value  $\tilde{\omega}$  without getting out of the JF Brans-Dicke theory. The value  $\omega = \tilde{\omega} = -3/2$  is a fixed point of the transformation (27). Any value of  $\omega \leq -3/2$  can be moved to any other value  $\tilde{\omega} \leq -3/2$  by choosing the gauge parameter  $\xi$ . However, it is not possible to move any value of  $\omega < -3/2$  across the divide  $\omega = \tilde{\omega} = -3/2$  to any other value  $\tilde{\omega} > -3/2$  and vice versa because the gauge parameter  $\xi$  becomes complex.

## V. Summary

The present article has been induced by the two articles [14], [15] that deal with the more general problem of  $f(R)$  gravity and general scalar-tensor theories. We have been concerned here only with the special case of  $\omega = \text{constant}$  vacuum Brans-Dicke theory and limited our analysis to the critical remarks against the range  $(-3/2 < \omega < -4/3)$  in [15, pp.2,7]. We agree that this range does not provide the ghost kinetic term via the mapping (4), hence wormhole, *in* EF and acknowledge the well known singularity problem in the Class I solution in JF.

We have argued that the two frames, JF and EF, are physically distinct and distinguishable; exoticity in one frame need not lead to exoticity in the other. Because of this, it makes sense to search for regular wormholes working only within any one of the frames. The old range  $(-3/2 < \omega < -4/3)$  in the Class I solution did yield some of the wormhole features except regularity [13]. In fact, the solution represents two separate singular space-times, not two regions of one *connected* space-time required of a true wormhole. Our principle has been, both

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<sup>5</sup>The prevailing belief has been that the Brans-Dicke scalar field  $\sigma$  possesses the asymptotic behavior  $\sigma = \sigma_0 + O(\frac{1}{\omega})$  so that one recovers general relativity in the limit  $\tilde{\omega} \rightarrow \infty$ . This is not the case; it instead shows the asymptotic behaviour  $\sigma = \sigma_0 + O(\frac{1}{\sqrt{\omega}})$ . To see it explicitly, take any starting finite value  $\omega < -3/2$  and expand  $\xi \simeq \frac{1}{2} \left( 1 \pm \frac{\sqrt{9+6\tilde{\omega}+6\omega+4\omega\tilde{\omega}}}{3+2\tilde{\omega}} \right)$ . To ensure the reality of  $\xi$ , we would require that  $\tilde{\omega} < -3/2$  as well. When  $\xi \rightarrow 1/2$ ,  $\tilde{\omega} \rightarrow -\infty$ , we have  $\sigma \simeq 1 \mp \sqrt{\frac{3+2\omega}{2\omega}} \ln \varphi$  so that  $\sigma_{,\mu} \simeq \mp \sqrt{\frac{3+2\omega}{2\omega}} (\ln \varphi)_{,\mu}$ . Then the first term on the right hand side of (2) does not vanish in the limit  $\tilde{\omega} \rightarrow -\infty$ . This is the source of trouble.



in [13] and here, to work solely within the JF without needing any mapping to EF. Accordingly, in the present article, we looked for a regular Brans-Dicke wormhole with two asymptotically flat regions of one connected spacetime and found it in (15)-(19) from Class I solution via operations (13). All the features of this wormhole can be readily transferred to EF, if necessary.

We have found that the operations (13) on the Brans Class I solution do remove the singularity in it. The transformed solution can be recognized as Brans Class II solution, which is the JF counterpart of the well known regular Ellis III wormhole in EF. As we see, the Brans Classes of solutions I and II are not independent, which seems to be a less known fact. Parallel to it, we show in Appendix that the Ellis I and III solutions in the minimally coupled theory are also not independent [23]. Our main conclusion is that the *transformed* Brans Class I solution (15)-(19) is a regular, twice asymptotically flat wormhole with all other desirable properties in the range  $-2 < \omega < -3/2$ . This incidentally answers the remark in [15] about this interval.

The conformal invariance of the vacuum JF theory shows that the solutions in it are not unique. Varying the real gauge parameter  $\xi$ , one can obtain any value  $\tilde{\omega}$  from a given  $\omega$  on either side of the divide  $\omega = -3/2$  but not *across* it since it is the fixed point of the transformation (27). Thus our new range  $-2 < \omega < -3/2$  is based on the gauge  $\xi = 0$ , while the lower limit ( $-2 < \omega$ ) can always be shifted to any value  $\tilde{\omega}$  for suitable  $\xi \neq 0$ . We confirm that the wormhole analogue of naked black holes exist also in the regular wormhole spacetime, as evident from figures 1 and 2.

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### Appendix

The following solution follows directly from Brans Class I solution under conformal transformation [4]:

$$d\tau_E^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu = - \left( \frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}} \right)^{2\beta} dt^2 + \left( 1 - \frac{m}{2r} \right)^{2(1-\beta)} \left( 1 + \frac{m}{2r} \right)^{2(1+\beta)} \times [dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\psi^2] \quad (\text{A1})$$

$$\phi(r) = \sqrt{\frac{2(1-\beta^2)}{\alpha}} \ln \left[ \frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}} \right], \quad (\text{A2})$$

of where  $\alpha$ ,  $m$  and  $\beta$  are arbitrary positive constants. This is known as Buchdahl solution [24] of Einstein minimally coupled theory (EF), rediscovered later as Ellis I solution. The metric is invariant in form under inversion (for integer  $\beta$ )

of the radial coordinate  $r \rightarrow \frac{m^2}{4r}$  and we have two asymptotically flat regions (at  $r = 0$  and  $r = \infty$ ), the minimum area radius (throat) occurring at  $r_0 = \frac{m}{2} [\beta + \sqrt{\beta^2 - 1}]$ . The real throat is guaranteed by the condition  $\alpha = -1$ ,  $\beta^2 > 1$ , leading to negative kinetic term. Once again, for  $\beta = 1$ , it reduces to the Schwarzschild black hole solution in isotropic coordinates and for  $\beta^2 > 1$ , it represents a naked singularity at  $r = m/2$ .

In the above solution, ghost matter exists as it can be easily seen that NEC and WEC are violated. However, despite having two flat asymptotic regions, the solution is spoilt because of the occurrence of singularity, though a traveler would never meet it.

Using the coordinate transformation  $l = r + \frac{m^2}{4r}$ , the solution (A1) and (A2) can be expressed as

$$ds^2 = -f_0(l)dt^2 + \frac{1}{f_0(l)} [dl^2 + (l^2 - m^2) (d\theta^2 + \sin^2 \theta d\psi^2)], \quad (\text{A3})$$

$$f_0(l) = \left( \frac{l-m}{l+m} \right)^\beta, \quad (\text{A4})$$

$$\phi_0(l) = \sqrt{\frac{\beta^2 - 1}{2}} \ln \left[ \frac{l-m}{l+m} \right]. \quad (\text{A5})$$

In the solution set  $(f_0, \phi_0)$ , we choose

$$m \rightarrow -im, \beta \rightarrow i\beta \quad (\text{A6})$$

so that the throat  $l_0 = m\beta$  remains invariant in sign and magnitude. Then the metric resulting from (A3) is

$$ds^2 = -f'_0(l)dt^2 + \frac{1}{f'_0(l)} [dl^2 + (l^2 + m^2) (d\theta^2 + \sin^2 \theta d\psi^2)] \quad (\text{A7})$$

$$f'_0(l) = \exp \left[ -2\beta \cot^{-1} \left( \frac{l}{m} \right) \right] \quad (\text{A8})$$

$$\phi'_0(l) = \left[ \sqrt{2} \sqrt{1 + \beta^2} \right] \cot^{-1} \left( \frac{l}{m} \right). \quad (\text{A9})$$

Using the relation

$$\cot^{-1}(x) + \tan^{-1}(x) = +\frac{\pi}{2}; x > 0 \quad (\text{A10})$$

$$= -\frac{\pi}{2}; x < 0 \quad (\text{A11})$$

we get from (A8), (A9)

$$f_{0\pm}(l) = \exp \left[ -2\beta \left\{ \pm \frac{\pi}{2} - \tan^{-1} \left( \frac{l}{m} \right) \right\} \right] \quad (\text{A12})$$

$$\phi_{0\pm}(l) = \left[ \sqrt{2}\sqrt{1+\beta^2} \right] \left[ \pm \frac{\pi}{2} - \tan^{-1} \left( \frac{l}{m} \right) \right] \quad (\text{A13})$$

We might study the solutions (A8) and (A9) *per se*, while allowing for a discontinuity at the origin  $l = 0$ . Alternatively, we might treat *each* of the  $\pm$  set in Eqs.(A12), (A13) as independently derived exact solution valid in the unrestricted range of  $l$  with no discontinuous jump at  $l = 0$ . The two alternatives do not appear quite the same. In fact, each of the individual branch represents a geodesically complete, asymptotically flat asymmetric wormhole having different masses, one positive ( $m\beta$ ) and the other negative ( $-m\beta e^{-\beta\pi}$ ), on two sides respectively. The commonly known Ellis III solution is the *+ve* branch which is continuous over the entire interval  $l \in (-\infty, +\infty)$ . The *-ve* branch also possesses exactly the same properties. What we have shown here is that the Ellis I and III solutions are *not* independent solutions of the Einstein minimally coupled scalar field theory. The Brans Class II wormhole is the JF counterpart of, but not the same as, Ellis III wormhole<sup>6</sup> since conformal mapping and Wick rotations do not commute.

#### Figure captions

Fig.1. We take  $B' = 1$ , the positive sign before  $\Lambda$  and a value in the new range  $-2 < \omega < -3/2$ , say,  $\omega = -1.7$ . The positive value of the throat radius is  $r_0^{'+} = 0.17$ ,  $C = -3.33$  and  $\Lambda = 0.81$ . The maximum value of  $\mathbf{R}_{\hat{1}\hat{\phi}\hat{1}\hat{\phi}}$  occurs at  $r' = 0.54$ , which lies above the throat  $r_0^{'+}$ . The plot shows curvature enhancement above the throat.

Fig.2. We take  $B' = 1$ , the negative sign before  $\Lambda$  and a value in the new range  $-2 < \omega < -3/2$ , say,  $\omega = -1.7$ . The positive value of the throat radius is  $r_0^{'+} = 5.88$ ,  $C = -3.33$  and  $\Lambda = -0.81$ . The minimum value of  $\mathbf{R}_{\hat{1}\hat{\phi}\hat{1}\hat{\phi}}$  occurs at  $r' = 8.60$ , which lies above the throat  $r_0^{'+}$ . The plot shows curvature depletion above the throat.

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<sup>6</sup>This is referred to also as Ellis-Bronnikov solution in [23]. The solutions of Einstein minimally coupled theory was first found by Fisher [25] in 1948 in a certain form, rediscovered by several authors afterwards differing only in coordinate choices. The notable authors, among others, include Buchdahl [24], Ellis [26] and Bronnikov [27]. We have shown that two classes of solutions can be derived from one another and are not really independent solutions.

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